Thematic Melodies of
20th Century Theoretical Physics
Assignments

As the schedule of the course will be quite compressed, students should do some preparatory work ahead of time, including some of the more detailed computations to be covered in the lectures later.

Academic Honesty

In doing these assignments, students should be aware of the University's policy on academic honesty; please see

`http://www.cuhk.edu.hk/policy/academichonesty/index.htm`

In particular, students must make an explicit declaration on each assignment for any problem that is (a) done in collaboration with others (state own share of effort), or (b) substantially taken from an existing source. Failure to do so will be considered a breach of academic honesty and may result in disciplinary action.

<u>Thermodynamics of black body radiation</u>

1. The historical development of quantum physics relied heavily on black-body radiation (BBR). In the lecture, it is mentioned that for BBR,

$$E = TS - PV \tag{1}$$

This problem shows that (1) is in general impossible, but is correct for BBR.

(a) Normally, the Gibbs' free energy of a system is defined by $G = E - TS + PV$. Show that $dG = -SdT + VdP$, and hence $S = -(\partial G/\partial T)_P$, $V = (\partial G/\partial P)_T$.

(b) If (1) is true, show that $G = S = V = E = 0$, which can only be true of a trivial system. Thus we (seem to) prove that (1) cannot be true.

(c) Now turn to BBR in particular. Since the energy density of BBR goes as T^4, $E = aT^4V$, where a is some constant.

(d) Since the momentum and the energy of a photon are the same (in units where $c = 1$), show that P is 1/3 of the energy density; hence $P = (a/3)T^4$. Hint: Similar to usual argument in the kinetic theory of gases.

(e) Put these results in the First Law of Thermodynamics: $dE = TdS - PdV$, and hence show that $dS = (4a/3)(3T^2V\,dT + T^3dV)$. Integrate this to show $S = (4a/3)T^3V$. Hence prove (1) for radiation.

(f) We have shown from general principles that (1) cannot be true, and yet explicitly verified that it is true for BBR. Resolve the paradox.

Least Action Principle

For the next few problems, you have to first learn: the least-action principle, the Lagrangian formulation of mechanics and the Hamiltonian formulation of mechanics.

2. A particle of mass m moves in a plane, with position described by the radius r and the angle ϕ. It is subject to a central force due to a potential $V(r)$. Write the Lagrangian L in terms of $r, \phi, \dot{r}, \dot{\phi}$. Hence write out the Euler–Lagrange equation for

$$m\frac{d^2r}{dt^2} = ?? \tag{2}$$

and identify the centrifugal force. This problem illustrates that the Newtonian equation of motion $m\,d^2x/dt^2 = -\partial V/\partial x$ cannot be generalized in the same form to non-cartesian coordinates, but the Euler–Lagrange equation can be so generalized.

3. Consider a relativistic free particle. For a small segment of motion covering $(\Delta t, \Delta x, \Delta y, \Delta z)$, the only invariant quantity that is linear in the size of the interval is $\Delta s = \sqrt{(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2}$, hence the action must take the form

$$\Delta S = k\sqrt{(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2} \tag{3}$$

for some constant k.

(a) Since $S = \int L\,dt$, show that $L = k\sqrt{1 - v^2}$, where $v = |\mathbf{v}|$ and $\mathbf{v}$ is the velocity.

(b) By expanding for small v and comparing with the non-relativistic case, express k in terms of the mass m.

(c) Find the canonical momentum $\mathbf{p} = \partial L/\partial \mathbf{v}$ and the energy as given by the Hamiltonian $H = \mathbf{v} \cdot \mathbf{p} - L$.

4. The action for a charged particle is given by

$$S = -m \int \sqrt{1 - v^2}\, dt + e \int A_\mu dx^\mu \tag{4}$$

For the first term see the last problem. The second term is the natural form involving a 4-vector potential $A^\mu = (\phi, \mathbf{A})$ in terms of the usual scalar potential ϕ and the vector potential $\mathbf{A}$, and $dx^\mu = (dt, dx, dy, dz) = dt\,(1, \mathbf{v})$. (Note $a_\mu b^\mu = -a_0 b_0 + \mathbf{a} \cdot \mathbf{b}$ and $a_0 = -a^0$.) This problem shows how the "strange" form of the Lorentz force law comes from the "natural" $eA_\mu dx^\mu$.

(a) Writing all terms as $S = \int L\, dt$, show that

$$L = -m\sqrt{1 - v^2} - e\phi + e\mathbf{v} \cdot \mathbf{A} \tag{5}$$

(b) Start from (5) and use the Euler–Lagrange equation of motion to derive the Lorentz force law.

5. Following from the above problem, derive the Hamiltonian H for a charged particle. For simplicity write out the result only for the non-relativistic case. The answer should be

$$H = \text{const} + \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\phi \tag{6}$$

where $\mathbf{p}$ is the canonical momentum (not the same as $m\mathbf{v}$). The combination $\mathbf{p} - e\mathbf{A}$ will be crucial in quantum theory, and in particular in gauge theory.

6. Use the above result for H and write out the Hamiltonian equations of motion. Show that these agree with the Lorentz force law.

Minimum principle

Many laws of physics originally expressed as differential equations can be recast into minimum principles. The next few problems give some simple examples, including a discrete example which contains some elements of the ideas needed when dealing with lattice gauge theories, e.g., defining derivatives (and hence the connection) on links.

7. Suppose we want to solve for a field $\Phi(x)$ that satisfies $\nabla^2\Phi(x) = 0$ for $x \in R$, where R is a region in space (for simplicity say in 2D), with the Dirichlet boundary condition $\Phi(x) = \phi_s(x)$ on the boundary ∂R, where ϕ_s is given. Show that this problem is equivalent to seeking the minimum of

$$S = \tfrac{1}{2}\int_R |\nabla\Phi(x)|^2 \, d^2x \tag{7}$$

over all Φ that satisfy the boundary condition. This equivalence proves the solution exists: since S is bounded from below, a minimum must exist.

8. The discrete version of the above problem is as follows. Let $\Phi(x)$ be defined an a lattice, say a square lattice of side a; thus $x \mapsto (i, j)$. We require Φ to satisfy

$$\nabla^2\Phi(i,j) \equiv a^{-2}\left[\Phi(i{+}1,j) + \Phi(i{-}1,j) + \Phi(i,j{+}1) + \Phi(i,j{-}1) - 4\Phi(i,j)\right] = 0 \tag{8}$$

in a region R and to have prescribed values ϕ_s on ∂R, which are the sites on the boundary.

   (a) Show that (8) is the sensible definition of the discrete laplacian.

   (b) Show that this problem is equivalent to seeking the minimum of

$$S = \frac{1}{2}\sum_{\rm nn}\left[\frac{1}{a}|\Phi(i_1,j_1) - \Phi(i_2,j_2)|\right]^2 \tag{9}$$

   where nn means that the sum is restricted to the sites (i_1, j_1) and (i_2, j_2) being nearest neighbors; thus each term corresponds to a *link*.

   (c) Show that the square bracket in (9) is the sensible definition of the discrete derivative along the link.

9. This problem continues from the last one and develops a numerical method (called the *relaxation method*) for solving such problems by iteration. Start with an initial guess for $\Phi(x)$ that satisfies the boundary condition. Then choose an internal site (i.e., not on ∂R), and replace $\Phi(x)$ at that site by the average value at the 4 nearest neighbor sites.

   (a) Show that this process causes S to decrease. (In fact, if $\Phi(x)$ can only be changed at this one site, this process makes S the smallest possible.) Hence it is plausible that this process, when iterated, will eventually converge to the correct field.

(b) As an example, let R consist of 4×4 sites on a square lattice, and suppose the boundary values of Φ (starting at one corner and going round the boundary) are $1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 2, 1$ (thus there is symmetry along a diagonal). $\Phi(x)$ is to be determined at the 4 internal sites. Find the values by the above method. (For this choice of boundary values, the solution should be obvious, and serves as a check. The method, which can readily be implemented on a spreadsheet, should however be independent of the boundary values.)

From Path Integral to Schrödinger's Equation

10. Gaussian integrals. All integrals are from $-\infty$ to ∞.

(a) Calculate $I_0 = \int e^{-x^2} dx$. Hint: Multiply two such integrals with dx and dy and convert to polar coordinates.

(b) Hence calculate $I_0(\epsilon) = \int e^{-x^2/\epsilon}\, dx$.

(c) Hence calculate $I_2(\epsilon) = \int x^2 e^{-x^2/\epsilon}\, dx$. Hint: $(d/d\alpha) I_0(\alpha^{-1})$.

11. This problem leads to formulas that are needed in deriving the Schrödinger equation from the path integral.

(a) Consider the integral

$$g(x) = \int e^{-(x-y)^2/\epsilon} f(y)\, dy \tag{10}$$

in the limit $\epsilon \to 0$. The gaussian is sharply peaked at $y = x$, so expand

$$f(y) = f(x) + u f'(x) + \tfrac{1}{2} u^2 f''(x) + \ldots \tag{11}$$

where $u = y - x$. Use the result of the last problem to derive the saddle point approximation

$$\int e^{-(x-y)^2/\epsilon} f(y)\, dy = \sqrt{\pi\epsilon} \left[f(x) + \frac{\epsilon}{4} f''(x) + \ldots \right] \tag{12}$$

(b) Also derive the stationary phase approximation

$$\int e^{-i(x-y)^2/\epsilon} f(y)\, dy = \sqrt{i\pi\epsilon} \left[f(x) + \frac{i\epsilon}{4} f''(x) + \ldots \right] \tag{13}$$

Translation, Rotation, Angular Momentum

12. Show that $\psi(x-\xi) = e^{i\xi p_x/\hbar}\psi(x)$, where $p_x = i\hbar(\partial/\partial x)$ is the momentum operator. Hint: Taylor series in ξ.

13. Suppose an operator $g = e^{i\xi T}$ is unitary for any real ξ: $g^\dagger g = I$. Show that T is self-adjoint: $T = T^\dagger$. (Actually it is enough if g is unitary for *one* non-zero real value of ξ, but for our application, this weaker form of the theorem is sufficient.)

14. Suppose $g(\xi) = e^{i\xi p_x/\hbar}$ and the Hamiltonian $H = (\hbar^2/2m)p_x^2 + V(x)$ satisfy $g(\xi)^{-1}Hg(\xi) = H$ for all real ξ. Show that $V(x)$ is a constant.

Rotation Group

15. Let $\mathbf{J} = \mathbf{r}\times\mathbf{p}$, where $\mathbf{p} = i\hbar\nabla$. Calculate $[J_a, J_b]$, $[J_a, r_b]$, $[J_a, p_b]$, $[J_a, J^2]$.

16. Consider the 3-dimensional space S spanned by the functions x, y, z (or more generally these functions multiplied by a function $f(r)$).

    (a) Show that this space is a *representation* of the rotation group in the sense that it is *closed* under $\mathbf{J}$: if $\psi \in S$, then $J_a\,\psi \in S$.

    (b) There is a linear combination

    $$\psi(\mathbf{r}) = (\beta_x x + \beta_y y + \beta_z z)f(r) \tag{14}$$

    that has $m = 0$, namely

    $$J_z\,\psi = i\hbar(x\partial_y - y\partial_x)\,\psi = 0 \tag{15}$$

    Find β_x, β_y, β_z up to normalization.

    (c) Likewise find the linear combination that has $m = 1$, namely

    $$J_z\,\psi = i\hbar(x\partial_y - y\partial_x)\,\psi = 1\cdot\hbar\,\psi \tag{16}$$

17. Consider the 6-dimensional space S spanned by the functions x^2, y^2, z^2, yz, zx, xy (or more generally these functions multiplied by a function $f(r)$).

    (a) Show that this space is a *representation* of the rotation group in the sense that it is *closed* under $\mathbf{J}$.

(b) Show that this is not an *irreducible* representation, in the sense there is a proper subspace S' that is closed under $\mathbf{J}$.

(c) Find the state in S (up to normalization) that has $m = 2$.

<u>Clebsch–Gordan coefficients and selection rules</u>

The subject of CG coefficients and selection rules is a large one, and the following problems illustrate the main ideas in simple examples.

18. Show that different angular momentum states are orthogonal, i.e., $\langle j', m' | j, m \rangle = 0$ unless $j' = j$ and $m' = m$. Hint: Consider $\langle j', m' | O | j, m \rangle$ for the operator $O = J_z, J^2$. Because O is hermitian, it can operate to either the left or the right.

19. It is shown in the lecture that

$$J_\pm |j, m\rangle = C_\pm(j, m) |j, m \pm 1\rangle \qquad , \qquad J_z |j, m\rangle = m |j, m\rangle \tag{17}$$

where

$$C_\pm(j, m) = \sqrt{j(j+1) - m(m\pm 1)} \tag{18}$$

Show that the *operators* r_m

$$r_1 = -(x + iy)/\sqrt{2} \qquad , \qquad r_0 = z \qquad r_{-1} = (x - iy)/\sqrt{2} \tag{19}$$

transform like $j = 1$ in the sense (compare (17))

$$[J_\pm, r_m] = C_\pm(1, m) r_{m\pm 1} \qquad , \qquad [J_z, r_m] = m r_m \tag{20}$$

Actually the same is true of any vector, not just $\mathbf{r}$.

20. Operate with $r_{m'}$ on states $|j, m\rangle$, and consider

$$\psi(m', m) \equiv r_{m'} |j, m\rangle \tag{21}$$

where the label j is omitted in ψ. The state ψ can be decomposed into eigenstates of angular momentum $\Psi(J, M) \equiv |J, M\rangle$. It is usual to write the reverse relation:

$$\Psi(J, M) = N \sum_{m', m} A(J, M; 1, m'; j, m) \psi(m', m) \tag{22}$$

where N is an overall normalization constant, and A is a CG coefficient that expresses how angular momenta j, m and $1, m'$ combine to give J, M. We illustrate the evaluation of A in one example.

(a) First show that $A = 0$ unless $M = m'+m$. Hint: Show that $J_z\, \psi(m', m) = (m'+m)\, \psi(m', m)$.

(b) Next show that the set of states $\psi(m', m)$ is closed under the angular momentum operators. Hint: Operate with some $J_{m''}$ and commute it with $r_{m'}$. Hence the space spanned by $\psi(m', m)$ must be a representation (not necessarily irreducible) of the angular momentum algebra. If one state $\Psi(J, M)$ is in this space, then $\Psi(J, M')$ must also be in this space, for all $M' = -J, \ldots, J$.

(c) In the rest of this problem, take $j = 2$. So there are 5 choices of m (from -2 to 2) and 3 choices of m' (from -1 to 1). It is convenient to draw these as a lattice of 5×3 points; the diagonal lines contain states of the same $M = m'+m$. We first deal with the upper right corner of this lattice. Because of the result in Problem 20a, the maximum M is 3, so the maximum J is also 3. (If there is a $J = 4$ state, then there must be an $M = 4$ state, which is impossible.) So

$$\Psi(3, 3) = N\psi(1, 2) \tag{23}$$

i.e., (up to a phase convention) $A(3, 3; 1, 1; 2, 2) = 1$.

(d) Now operate on (23) with J_-. Note: This moves to the next diagonal. The calculation is shown schematically below, with X denoting (different) numerical constants that you should fill in:

$$\begin{aligned} J_-\Psi(3, 3) &= X\Psi(3, 2) \\ J_-\psi(1, 2) &= J_- r_1 |j{=}2, m{=}2\rangle \\ &= [J_-, r_1]|j{=}2, m{=}2\rangle + r_1 J_- |j{=}2, m{=}2\rangle \\ &= X r_0 |j{=}2, m{=}2\rangle + r_1 X |j{=}2, m{=}1\rangle \\ &= X\psi(0, 2) + X\psi(1, 1) \end{aligned} \tag{24}$$

Hence express $\Psi(3, 2)$ in terms of $\psi(0, 2)$ and $\psi(1, 1)$ and thus obtain $A(3, 2; 1, 0; 2, 2)$ and $A(3, 2; 1, 1; 2, 1)$.

(e) Continue and express $\Psi(3,1)$ in terms of $\psi(m',m)$ (where $m'+m = 1$), and likewise $\Psi(3,0)$, $\Psi(3,-1)$, $\Psi(3,-2)$, $\Psi(3,-3)$.

(f) Consider the space spanned by $\psi(0,2)$ and $\psi(1,1)$, i.e., the second diagonal. In Problem 20d one linear combination has been identified as $\Psi(3,2)$. Construct the other linear combination that is orthogonal. This must be $\Psi(2,2)$ (because of the states left, the maximum M is 2). Then use J_- as before to construct the other $\Psi(2,M)$.

(g) Again use orthogonality to construct $\Psi(1,1)$ and then J_- to construct the other $\Psi(1,M)$.

(h) Show that this exhausts all the states. This can be checked by counting: $J=3$ contains 7 states; $J=2$ contains 5 states and $J=1$ contains 3 states; $7+5+3=15$. Also convince yourself that in general $r_{m'}|j,m\rangle$ is a linear combination of $\psi(J,M)$ with allowed values of J being $j+1, j, j-1$. At least check the number of states is consistent.

21. The energy of an electric dipole in the presence of an external field $\mathbf{E}$ is $e\mathbf{E}\cdot\mathbf{r}$. Hence the dipole transition matrix element between two states is proportional to

$$\langle j',m'|e\mathbf{E}\cdot\mathbf{r}|j,m\rangle \tag{25}$$

Using the result of the last problem, show that this vanishes unless $|j-j'|\leq 1$. This is an example of a *selection rule*.

Groups and representations

22. The special unitary group SU(N) consists of $N\times N$ matrices g that are *unitary* in the sense $g^\dagger g = I$ and *special* in the sense that $\det g = 1$.

(a) Any square matrix g can be written as an exponential: $g = e^A$, where A is another matrix of the same size. Show that any unitary matrix can be written as $g = e^{iT}$ where T is hermitian ($T^\dagger = T$). (See Problem 13.)

(b) If $\det g = 1$, show that tr $T = 0$. Hint: Use the identity $\det(\exp A) = \exp(\text{tr } A)$, readily proved if A can be diagonalized — which is the case if A is hermitian or anti-hermitian.

(c) The collection of hermitian traceless $N \times N$ matrices forms a vector space of dimension $M = N^2 - 1$. Choose a basis T_a, $a = 1, \ldots, M$. Make this orthonormal under the inner product $(T, T') = \mathrm{tr}\,(TT')$. Thus we may assume

$$\mathrm{tr}\,(T_a T_b) = \delta_{ab} \tag{26}$$

Therefore show that SU(N) can be regarded as the collection of all e^{iT}, where $T = \theta^a T_a$, and θ^a are real parameters.

23. Consider their commutators, say $[T_a, T_b]$.

(a) The commutator is certainly an $N \times N$ matrix. Show that it is also traceless. Thus, it can be expressed as a linear combination

$$[T_a, T_b] = i f_{abc} T_c \qquad \text{(sum over } c \text{ implied)} \tag{27}$$

thus defining the *structure constants* f_{abc}.

(b) Show that f_{abc} are real. (This is the reason for inserting the factor i in (27)). Hint: T_a are hermitian.

(c) Show that f_{abc} is cyclic in the three indices, i.e., $f_{abc} = f_{bca} = f_{cab}$. Hint: Take a suitable trace.

(d) Show that f_{abc} is totally antisymmetric in the three indices (just like the Levi-Civita symbol). Hint: Because of the cyclic property, it suffices to show antisymmetry between the first two indices, for example.

(e) Show that the product of two f's satisfies the identity

$$f_{abd} f_{dce} + f_{bcd} f_{dae} + f_{cad} f_{dbe} = 0 \tag{28}$$

Hint: Jacobi identity (See Problem 31a.) Note: the indices in (28) go as: 1, 2, 5 cyclic; 3, 4 summed; 6 fixed.

In fact, any set of operators T_a that satisfy the same commutation relations would form the same algebra, and the corresponding g's would form the same group. The $N \times N$ matrices described above constitute the simplest example, or specifically the *fundamental representation.*

24. For SU(2), the standard choice for the fundamental representation is $T_a = (1/2)\sigma_a$, where σ_a are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{29}$$

Show that the T_a's are orthonormal in the sense (26), and calculate the structure constants f_{abc} in this case.

25. For SU(3), the standard choice for the fundamental representation is $T_a = (1/2)\lambda_a$, where λ_a are the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{30}$$

Again show that the T_a's are orthonormal in the sense (26), and calculate the structure constants f_{abc} in this case.

26. This problem deals with more than one representation, so we take more care in distinguishing abstract operators (e.g., $\hat{T}$) from matrices (e.g. $[T]$ with elements $[T]_{ij}$).

   (a) Suppose particles $|i\rangle$ constitute a representation of the group in the sense

$$\hat{T}_a|i\rangle = [T_a]_{ij}|j\rangle \tag{31}$$

   The antiparticles $|\bar{i}\rangle$ have opposite quantum numbers, so one may expect that, e.g.,

$$\hat{T}_a|\bar{i}\rangle = -[T_a]_{ij}|\bar{j}\rangle \tag{32}$$

Show that this does *not* give the correct commutation relation. Instead, consider the *conjugate representation* $[T'_a] = -[T_a]^*$, or more specifically when acting on the antiparticles

$$\hat{T}_a|\bar{i}\rangle = [T'_a]_{ij}|\bar{j}\rangle = -[T_a]^*_{ij}|\bar{j}\rangle \tag{33}$$

Show that this gives the correct commutation relation. The quantum numbers are related to the matrix elements of the diagonal T's, so indeed have opposite values.

(b) Show that the state $|S\rangle = \sum_i |i\rangle\,|\bar{i}\rangle$ is a *singlet* in the sense that $\hat{T}_a|S\rangle = 0$ for all T_a. Hint: The operators $\hat{T}$ are hermitian.

(c) Hence show that in SU(3) color theory, the following state is a singlet:

$$|S\rangle = |R\rangle\,|\bar{R}\rangle + |G\rangle\,|\bar{G}\rangle + |B\rangle\,|\bar{B}\rangle \tag{34}$$

27. Another representation of SU(N) is obtained by the matrices

$$[T_a]_{bc} = -if_{abc} \tag{35}$$

Since $b, c = 1, \ldots, M$, $M = N^2 - 1$, this is an M-dimensional representation, called the *adjoint representation*.

(a) Show that these matrices satisfy $[T_a, T_b] = if_{abc}T_c$. Hint: Take the $[\ ]_{de}$ element of the above relation; this will involve three terms, each with the product of two f's. Show that this equation is the same as (28).

(b) Write out these 3×3 matrices in the case of SU(2).

(c) These matrices are pure imaginary. Show that the associated group elements $g = \exp(i\theta^a T_a)$ are *orthogonal* matrices, i.e., real matrices satisfying $g^{\mathrm{T}}g = I$. Thus the g's can also be considered elements of the orthogonal group in M dimensions, $O(M)$. This leads to a natural correspondence between $SU(N)$ and $O(M)$.

Gauge Theory: EM

28. Consider the vector potentials $\mathbf{A} = (1/2)B_0\,(x\hat{\mathbf{j}} - y\hat{\mathbf{i}})$ and $\mathbf{A} = B_0\,x\hat{\mathbf{j}}$. Show that they correspond to the same magnetic field $\mathbf{B}$. In particular, show that the two vector potentials are related by $\mathbf{A} \mapsto \mathbf{A} - \nabla\Lambda(\mathbf{r})$ and find the gauge function Λ.

29. How can we guarantee that two such vector potentials, when used in the Schrödinger equation

$$\left[\frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + V\right]\psi = E\,\psi \tag{36}$$

will give the same energy E? Show that if we change both the vector potential and the wavefunction by

$$\mathbf{A} \mapsto \mathbf{A} - \nabla\Lambda(\mathbf{r}) \qquad , \qquad \psi \mapsto e^{i\theta(\mathbf{r})}\,\psi \tag{37}$$

then the Schrödinger equation remains valid with the same E, provided Λ and θ are related in a suitable way.

<u>Connection and curvature</u>

30. Consider a transformation on a vector wavefunction $\vec{\Psi}$

$$\vec{\Psi}(x) \mapsto \vec{\Psi}'(x) = e^{iT(x)}\vec{\Psi}(x) \tag{38}$$

where x denotes a point in some space and the position-dependent matrix $T(x)$ can be written more explicitly as $T(x) = \theta^a(x)T_a$, where $\theta^a(x)$ are position-dependent parameters and T_a are linearly independent matrices forming a basis. Note that (37) is a special case, where there is only one $T_a = I$. For simplicity we restrict to T being infinitesimal. It has been shown (see lectures) that the connection 1-form is then transformed by (to first order in T)

$$\Gamma_\mu \mapsto \Gamma'_\mu = \Gamma_\mu + i[T, \Gamma_\mu] - i\partial_\mu T \tag{39}$$

The last term, which is *not* linear in Γ_μ, arises because in a sense Γ_μ refers not to *one* point, but to two neighboring points. However, because the curvature tensor relates to only one point, it was claimed that

$$R_{\mu\nu} \mapsto R'_{\mu\nu} = e^{iT}R_{\mu\nu}e^{-iT} = R_{\mu\nu} + i[T, R_{\mu\nu}] \tag{40}$$

where the last form is valid to first order in T. This transformation law is linear in $R_{\mu\nu}$. Prove (40) explicitly by starting from (39) and the definition of the curvature tensor

$$R_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] \tag{41}$$

Hint: (a) work to first order in T; (b) use the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{42}$$

31. This problem proves the *Bianchi identity* for the curvature tensor.

(a) By writing out the terms, prove the Jacobi identity (42).

(b) Therefore we have

$$[D_\rho, [D_\mu, D_\nu]] + \text{cyclic} = 0 \tag{43}$$

(c) Next note that $[D_\mu, D_\nu] = R_{\mu\nu}$ is a multiplicative operator (i.e., a matrix) while D_ρ operating on a product behaves like a differential operator, so

$$\begin{aligned} [D_\rho, [D_\mu, D_\nu]]\, \vec{\Psi} &= [D_\rho, R_{\mu\nu}]\, \vec{\Psi} \\ &= D_\rho \left(R_{\mu\nu} \vec{\Psi} \right) - R_{\mu\nu} \left(D_\rho \vec{\Psi} \right) \\ &= \left(D_\rho R_{\mu\nu} \right) \vec{\Psi} \end{aligned} \tag{44}$$

so we have the operator equation

$$D_\rho R_{\mu\nu} + \text{cyclic} = 0 \tag{45}$$

which is known as the Bianchi identity. Often we use the notation $D_\rho X \equiv X_{;\rho}$, so

$$R_{\mu\nu;\rho} + \text{cyclic} = 0 \tag{46}$$

Finally, restoring the indices on the matrix $R_{\mu\nu}$

$$R^i{}_{j\mu\nu;\rho} + (\text{cyclic } \mu\nu\rho) = 0 \tag{47}$$

<u>Monopoles</u>

32. This problem considers the possibility of magnetic monopoles and gives a simple derivation of the quantization condition [1]. Let the static equations given by

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad , \qquad \nabla \cdot \mathbf{B} = 4\pi\rho' \tag{48}$$

where ρ is density of electric charge and ρ' is the density of magnetic charge. Let there be a point monopole of strength g, so that the magnetic field is $\mathbf{B} = g\mathbf{r}/r^3$ with $\mathbf{r}$ measured from the position of the monopole. Consider the head-on collision of such a monopole with a charge e at the origin, the latter regarded as a small sphere. Cut the spherical distribution of charge into many rings, and first consider a thin ring of radius r situated on the x-y plane, with center at the origin, and carrying a charge q. The monopole is on the z-axis, and travels along the z-axis, through the ring.

(a) Show that the rate of change of the angular momentum of the ring is given by (up to signs which we can be sloppy about)

$$\frac{dL_z}{dt} = Fr = qEr = \frac{1}{2\pi} q\mathcal{E} = \frac{q}{2\pi}\frac{d\Phi}{dt} \tag{49}$$

where F is the tangential force, E is the tangential electric field, $\mathcal{E}$ is the emf, and Φ is the magnetic flux.

(b) Hence, integrating over time, show that

$$\Delta L_z = \frac{q}{2\pi}\Delta\Phi = 2qg \tag{50}$$

since a total flux of $4\pi g$ passes through the surface enclosed by the ring as the monopole goes from $z = -\infty$ to $z = \infty$.

(c) Show that if there is a charge e in a spherical distribution, it can be regarded as the sum of many rings each with a charge q. Then for the charge e as a whole,

$$\Delta L_z = 2eg \tag{51}$$

(d) But in quantum mechanics, ΔL_z must be a multiple of $\hbar$. Hence show the quantization condition

$$eg = n\hbar/2 \tag{52}$$

(e) Go back to the case of a single ring, and sketch the flux through the ring as a function of time t, as the monopole travels along z; take flux towards $+z$ as positive. Actually the flux goes from 0 to $+2\pi g$, jumps to $-2\pi g$ and then goes to zero; so the total change in flux is 0, not $4\pi g$! Fix this problem by a suitable argument.

(f) Show that the ratio between the magnetic force (between elementary monopoles) and the electric force (between elementary charges) is given by $1/(4\alpha^2)$, where $\alpha \approx 1/137$ is the fine structure constant. Note: $\alpha = e^2/(\hbar c)$ and use $c = 1$. Hence verify that magnetic forces are "large".

General relativity

33. In general relativity, we consider vectors in tangent space, which has the same dimension as the underlying manifold, so the Bianchi identity (see Problem 31) takes the form $R^{\alpha}{}_{\beta\mu\nu;\rho} + (\text{cyclic } \mu\nu\rho) = 0$, where all Greek indices run over $0, 1, 2, 3$. Define

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \qquad , \qquad R = g^{\mu\nu} R_{\mu\nu} \qquad , \qquad G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} \tag{53}$$

and prove that the Einstein tensor $G_{\mu\nu}$ is conserved in the sense

$$G^{\mu\nu}{}_{;\nu} = 0 \tag{54}$$

Note that indices are raised by $g^{\cdot\cdot}$ and lowered by $g_{\cdot\cdot}$.

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[1] HM Lai and K Young, Phys. Rev. D28, 1552 (1983)