Quantization and generator of translation in generalized coordinates

K. Young*
Department of Physics, The Chinese University of Hong Kong, Hong Kong, People's Republic of China
(Dated: November 7, 2006)

It is shown that for any generalized coordinate X, the generator of translation in X is just the corresponding conjugate momentum P. This result is deeply related to a broader theorem, namely that quantum-mechanical canonical commutators or classical Poisson brackets are invariant with respect to coordinate transformations.

PACS numbers: 00.00

I. INTRODUCTION

In the lectures, we showed that if a rectangular coordinate x is displaced by ξ, i.e., $x \mapsto x-\xi$, the operator involved is $\exp(-i\xi p_x/\hbar)$, so that the generator of infinitesimal translations is p_x. We also showed that under a rotation about the z-axis, i.e., $\phi \mapsto \phi-\theta$, the corresponding operator is $\exp(-i\theta J_z/\hbar)$, so that the generator of infinitesimal rotations about z is J_z. It is noticed that J_z *happens to be* the conjugate momentum corresponding to ϕ, i.e., classically

$$J = \frac{\partial L}{\partial \dot{\phi}} \tag{1}$$

A student therefore asked the very important question (slightly paraphrased to make it more precise):

> How can we guarantee, in general, that when we displace a generalized coordinate X, the generator is the corresponding conjugate momentum P?

This set of notes address this question. The argument is given in three steps:

- First we show $[X, P] = i\hbar$.
- It then follows that $P = -i\hbar(\partial/\partial X)$.
- Then the generator for translation in X is just P.

The first two steps are relevant beyond translations. In fact, these correspond to the statement that canonical quantization is invariant with respect to coordinate transformations.

II. CANONICAL QUANTIZATION

A. The question

Consider a Lagrangian L written in terms of a set of coordinates x_i:

$$L = L(x_i, \dot{x}_i) \tag{2}$$

from which we define the conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \tag{3}$$

Quantization is achieved by imposing the canonical commutators

$$[x_i, p_j] = i\hbar\delta_{ij} \tag{4}$$

Thus p_j is represented by

$$p_j = -i\hbar\frac{\partial}{\partial x_j} \tag{5}$$

Now change to a set of generalized coordinates X_α, so that

$$L = L(X_\alpha, \dot{X}_\alpha) \tag{6}$$

from which we define the conjugate momenta

$$P_\alpha = \frac{\partial L}{\partial \dot{X}_\alpha} \tag{7}$$

Quantization can also be achieved by imposing the canonical commutators

$$[X_\alpha, P_\beta] = i\hbar\delta_{\alpha\beta} \tag{8}$$

How can we guarantee that (8) is consistent with (4)? In other words, we are only allowed to impose *one* quantization rule, say (4); can we then derived (8) as a consequence?

We shall prove these results for cases where the velocities appear in L quadratically

$$L = \frac{1}{2}m_{ij}(x)\dot{x}_i\dot{x}_j - V(x) \tag{9}$$

For this Lagrangian, (3) leads to

$$p_i = m_{ij}\dot{x}_j \tag{10}$$

*Electronic address: kyoung@cuhk.edu.hk

B. Commutator for generalized coordinate and momentum

First, we note that

$$\dot{x}_i = \frac{dx_i}{dt} = \frac{\partial x_i}{\partial X_\alpha}\frac{dX_\alpha}{dt} \equiv A_{i\alpha}\dot{X}_\alpha \tag{11}$$

so the Lagrangian can be written as

$$L = \frac{1}{2}M_{\alpha\beta}(X)\dot{X}_\alpha\dot{X}_\beta - V(X) \tag{12}$$

where

$$M = A^{\mathrm{T}}mA \tag{13}$$

Then it follows, in analogy with (10)

$$P_\alpha = M_{\alpha\beta}\dot{X}_\beta \tag{14}$$

But expressing $\dot{X}_\beta$ in terms of $\dot{x}_k$, we have, by using the inverse of (11),

$$\dot{X}_\beta = B_{\beta k}\dot{x}_k \tag{15}$$

where $B = A^{-1}$. Thus, putting (14) and (15) together,

$$\begin{aligned} P_\alpha &= M_{\alpha\beta}B_{\beta k}\dot{x}_k \\ &= M_{\alpha\beta}B_{\beta k}(m^{-1})_{kj}p_j \end{aligned} \tag{16}$$

where in the last step we have used the inverse of (10).

Then we have

$$\begin{aligned} [X_\gamma, P_\beta] &= [X_\gamma, M_{\alpha\beta}B_{\beta k}(m^{-1})_{kj}p_j] \\ &= M_{\alpha\beta}B_{\beta k}(m^{-1})_{kj}\,[X_\gamma, p_j] \end{aligned} \tag{17}$$

since the prefactor commutes with all coordinates. Then using the theorem

$$[f(x), p_j] = \frac{\partial f}{\partial x_i}\,[x_i, p_j] \tag{18}$$

we then have

$$\begin{aligned} [X_\gamma, P_\beta] &= \frac{\partial X_\gamma}{\partial x_i}\,M_{\alpha\beta}B_{\beta k}(m^{-1})_{kj}\,[x_i, p_j] \\ &= B_{\gamma i}M_{\alpha\beta}B_{\beta k}(m^{-1})_{kj}\,[x_i, p_j] \\ &= i\hbar\,B_{\gamma i}M_{\alpha\beta}B_{\beta k}(m^{-1})_{ki} \\ &= i\hbar\,\left\{MBm^{-1}B^{\mathrm{T}}\right\}_{\alpha\gamma} \end{aligned} \tag{19}$$

But

$$MBm^{-1}B^{\mathrm{T}} = (A^{\mathrm{T}}mA)Bm^{-1}B^{\mathrm{T}} = I \tag{20}$$

by using $AB = I$. This then proves (8), and shows that

$$P_\alpha = -i\hbar\frac{\partial}{\partial X_\alpha} \tag{21}$$

Although the above derivation is given in the language of quantum-mechanical commutators, it can be rephrased in terms of Poisson brackets, and is a special case of an important theorem in classical mechanics, namely

> Poisson brackets are invariant under contact transformations.

The general proof of this statement is given in most textbooks on classical mechanics.

III. TRANSLATION

Now consider a translation in the generalized coordinates: $X_\alpha \mapsto X_\alpha - \Xi_\alpha$. Clearly wavefunctions are transformed by the operator

$$\exp(-\Xi_\alpha(\partial/\partial X_\alpha)) \tag{22}$$

which is just an application of the Taylor expansion. But by (21), this then becomes

$$\exp(-i\Xi_\alpha P_\alpha/\hbar) \tag{23}$$

thus showing that the generator for translation of X_α is exactly P_α.